Memo, WTW 158, semestertest 2, 2007

Vraag 1 / Question 1

Bepaal die afgeleides van die volgende funksies:
(MOENIE JOU ANTWOORDE VEREENVOUDIG NIE!)

Find the derivatives of the following functions:
(DO NOT SIMPLIFY YOUR ANSWERS!):

i $f(x) = a^3 + \cos^3(x^2)$

$$f'(x) = 0 + 3\cos^2(x^2) \times -\sin x^2 \times 2x$$

ii $g(x) = e^{-5x}\sin 3x$

$$g'(x) =$$
$$e^{-5x} \times -5 \times \sin 3x + e^{-5x} \times \cos 3x \times 3$$

iii $y = 10^{1-x^2}$

$$\frac{dy}{dx} = \ln 10 \times 10^{1-x^2} \times -2x$$

iv $y = \dfrac{\cosh x}{\sin^{-1}x}$ (Notasie/Notation: $\sin^{-1}x = \text{bgsin}(x) = \arcsin(x)$)

$$\frac{dy}{dx} = \frac{\sinh x \times \sin^{-1}x - \cosh x \cdot \frac{1}{\sqrt{1-x^2}}}{(\sin^{-1}x)^2}$$

v $f(\theta) = \frac{d}{d\theta}(e^{3\theta} + \theta^{e})$

$= 3e^{3\theta} + e\theta^{e-1}$

$\therefore f'(\theta) = 3e^{3\theta} \times 3 + e(e-1)\theta^{e-2}$

vi $f(x) = \ln(x^2 + 6x)$

$f'(x) = \frac{1}{x^2+6x} \times (2x+6)$

vii $y = x^{\tan x}$

$\ln y = \ln(x^{\tan x}) = \tan x \times \ln x$

$\therefore \frac{1}{y}\frac{dy}{dx} = \sec^2 x \times \ln x + \tan x \times \frac{1}{x}$

$\therefore \frac{dy}{dx} = y\left[\sec^2 x \ln x + \tan x \times \frac{1}{x}\right]$

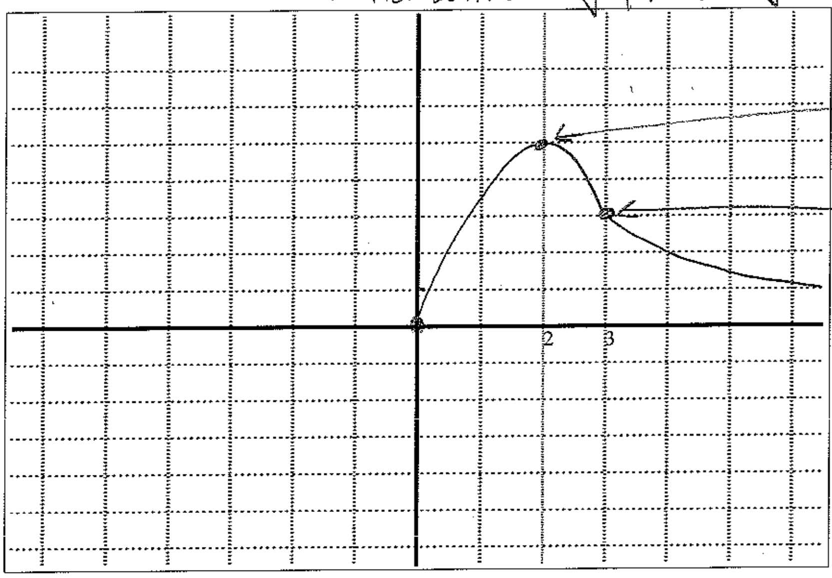
[15]

Vraag 2 / Question 2

Skets die grafiek van 'n funksie $y = f(x)$ wat al die voorwaardes hieronder bevredig.

Sketch the graph of a function $y = f(x)$ that satisfies all the conditions below.

1. $x \in R$
2. $f(0) = 0$
3. $f(-x) = -f(x)$ ⇒ odd function
4. $f'(x) > 0,\ |x| < 2$ ⇒ increasing on $(-2, 2)$
5. $f'(x) < 0,\ |x| > 2$ ⇒ decreasing on $(-\infty, -2)$ and $(2, \infty)$
6. $f''(x) < 0,\ 0 < x < 3$ ⇒ concave down on $(0, 3)$
7. $f''(x) > 0,\ x > 3$ ⇒ concave up on $(3, \infty)$
8. $f'(2) = 0$ ⇒ local max at $x = 2$
9. $\lim_{x \to \infty} f(x) = 1$ ⇒ horizontal asymptote $y = 1$

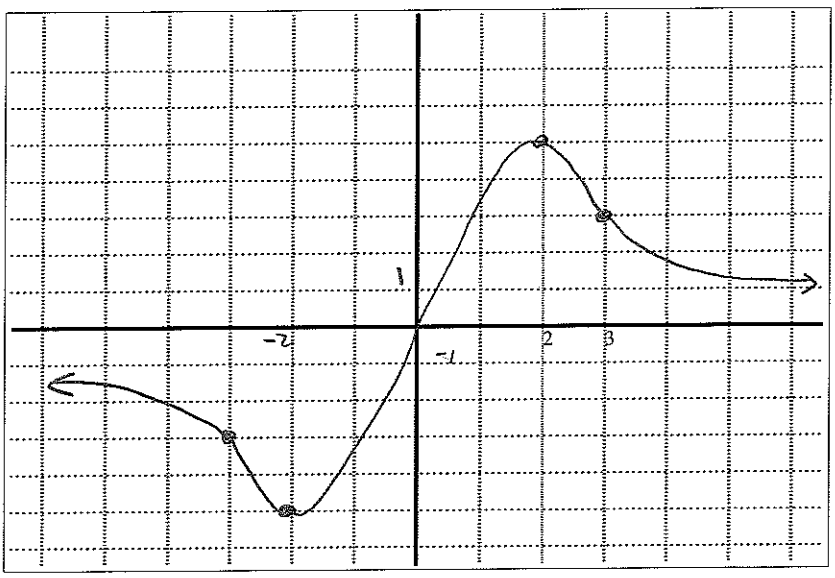


local maximum

inflection point

Rofwerk / Scribling

now use the fact that the function is odd



[3]

Vraag 3 / Question 3

(i) Bepaal die x –waardes van die punte op die kromme $y = \cos x + \cos^2 x$, $x \in [\frac{\pi}{2}, \frac{3\pi}{2}]$, waar die raaklyne horisontaal is.

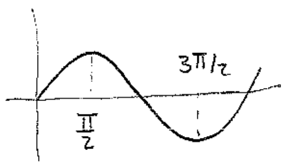
Find the x values of the points on the curve $y = \cos x + \cos^2 x$, $x \in [\frac{\pi}{2}, \frac{3\pi}{2}]$, where the tangent lines are horizontal.

$$\frac{dy}{dx} = -\sin x + 2\cos x(-\sin x)$$

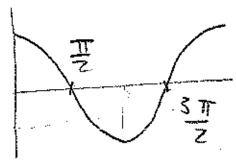
$$= -\sin x(1 + 2\cos x)$$

$$\frac{dy}{dx} = 0 \Rightarrow \sin x = 0 \quad \text{or} \quad \cos x = -\frac{1}{2}$$

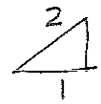
$$\Rightarrow \quad x = \pi \quad \text{or} \quad x = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \quad \text{or} \quad x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$$



$y = \sin x$



$y = \cos x$



[4]

(ii) Gee die vergelyking van enige een van die raaklyne in Vraag 3(i)

Give the equation of any one of the tangent lines in Question 3(i).

$y = mx + b = 0x + b = b$ because the slope, m, is zero.

If $x = \pi$ then $y = \cos\pi + \cos^2\pi = -1 + (-1)^2 = 0$ [1]

If $x = \frac{4\pi}{3}$ then $y = \cos\frac{4\pi}{3} + \cos^2(\frac{4\pi}{3}) = -\frac{1}{2} + (-\frac{1}{2})^2$

$= -\frac{1}{4}$

The equations are $y = 0$ and $y = -\frac{1}{4}$

Vraag 4 /Question 4

Gegee/Given $f(x) = \frac{1}{2}x^2 - \frac{2}{3}x^{\frac{3}{2}}, \ x > 0.$

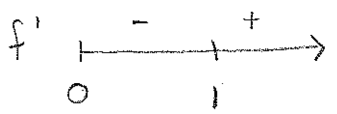
(i) Vind die interval(le), indien einge, waar die grafiek van f stygend is.

Find the interval(s), if any, on which the graph of f is increasing.

$f'(x) = \frac{1}{2} \times 2x - \frac{2}{3} \times \frac{3}{2} x^{\frac{1}{2}} = x - \sqrt{x} = \sqrt{x}(\sqrt{x} - 1)$

$f'(x) = 0 \Rightarrow \sqrt{x} = 0$ or $\sqrt{x} = 1 \Rightarrow x = 0$ or $x = 1$

The only critical number is $x = 1$ $(x > 0)$



$f'(x) > 0 \Rightarrow x > 1$

f is increasing on $(1, \infty)$

[2]

(ii) Gee, indien dit bestaan, die koördinate van die lokale maksimum punt(e) EN lokale minimum punt(e) van f.

Give, if they exist, the coordinates of the local maximum point(s) AND the local minimum point(s) of f.

from (i) the only critical number is $x = 1$

as f is decreasing on $(0,1)$ and increasing on $(1, \infty)$ it follows that f has a local minimum at $x = 1$

$f(1) = \frac{1}{2} - \frac{2}{3} = \frac{3-4}{6} = -\frac{1}{6}$

[2]

f has a local minimum at $(1, -\frac{1}{6})$

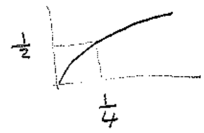
f has no local maximum

(iii) Vind die interval(le), indien enige, waar die grafiek van f konkaaf na bo is.

Find the interval(s), if any, on which the graph of f is concave up.

$$f''(x) = 1 - \frac{1}{2}x^{-\frac{1}{2}} = 1 - \frac{1}{2\sqrt{x}} = \frac{2\sqrt{x}-1}{2\sqrt{x}}$$

$$f''(x) > 0 \Rightarrow 2\sqrt{x} - 1 > 0 \Rightarrow \sqrt{x} > \frac{1}{2}$$



$$\Rightarrow x > \frac{1}{4}$$

$\therefore$ f is concave up on $(\frac{1}{4}, \infty)$

[2]

(iv) Bepaal $\lim_{x \to \infty} f(x)$. Toon alle stappe.

Find $\lim_{x \to \infty} f(x)$. Show all steps.

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1}{2}x^2 - \frac{2}{3}x^{3/2} \quad (\text{form } \infty - \infty)$$

$$= \lim_{x \to \infty} x^{3/2}\left(\frac{1}{2}x^{\frac{1}{2}} - \frac{2}{3}\right)$$

$$= +\infty$$

[1]

Vraag 5 / Question 5

Vind die absolute/globale maksimum en absolute/globale minimum van

$f(x) = \frac{\ln x}{x}$, $x \in [1,3]$.

Find the absolute/global maximum and absolute/global minimum of

$f(x) = \frac{\ln x}{x}$, $x \in [1,3]$.

$$f'(x) = \frac{\frac{1}{x} \times x - \ln x . 1}{x^2} = \frac{1 - \ln x}{x^2}$$

$$f'(x) = 0 \Rightarrow 1 - \ln x = 0 \Rightarrow \ln x = 1$$

$$\Rightarrow x = e^1$$

The only critical number is at $x = e$.

as f is continuous on the closed interval $[1,3]$ [3]

f has global extremes.

x	1	e	3
$f(x)$	0	$0{,}3678\ldots$ $= \frac{1}{e}$	$0{,}3662\ldots$

The absolute/global maximum has coördinates $(e, \frac{1}{e})$)

The absolute/global minimum has coördinats $(1,0)$

Vraag 6 / Question 6

(i) Bepaal $\lim_{x\to\infty} x^2e^{-3x}$. (Jy mag nie 'n tabel gebruik nie.)

Find $\lim_{x\to\infty} x^2e^{-3x}$ (You may not use a table.)

$$\lim_{x\to\infty} x^2e^{-3x} \quad (\text{form } \infty \times 0)$$

$$= \lim_{x\to\infty} \frac{x^2}{e^{3x}} \quad \left(\text{form } \frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{2x}{3e^{3x}} \quad \left(\text{form } \frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{2}{9e^{3x}}$$

$$= 0$$

(ii) Bepaal $\lim_{x\to-\infty} x^2e^{-3x}$. (Jy mag nie 'n tabel gebruik nie.)

Find $\lim_{x\to-\infty} x^2e^{-3x}$. (You may not use a table.)

$$\lim_{x\to-\infty} x^2e^{-3x}$$

$$= \infty$$

because $x^2 \to \infty$ if $x \to -\infty$

and

$e^{-3x} \to \infty$ if $x \to -\infty$

[3]

Vraag 7 / Question 7

Laat $y(x) = \text{bgsin}(x) = \sin^{-1}x$. Toon aan dat $y'(x) = \frac{1}{\sqrt{1-x^2}}$.

Let $y(x) = \arcsin(x) = \sin^{-1}x$. Show that $y'(x) = \frac{1}{\sqrt{1-x^2}}$.

$y(x) = \text{bgsin}\, x \iff \sin[y(x)] = x$ and $y(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$\Rightarrow \cos(y(x)) \times y'(x) = 1$$

$$\Rightarrow y'(x) = \frac{1}{\cos(y(x))}$$

$\cos^2(y(x)) + \sin^2(y(x)) = 1 \Rightarrow \cos^2(y(x)) = 1 - \sin^2(y(x))$
$= 1 - x^2$.

as $y(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ it follows that $\cos(y(x)) \geq 0$

and therefore $\cos(y(x)) = +\sqrt{1-x^2}$

$\therefore y'(x) = \frac{1}{\cos(y(x))} = \frac{1}{\sqrt{1-x^2}}$, $x \in (-1,1)$

[3]

Vraag 8 / Question 8

Formuleer die Middelwaardestelling

State the Mean Value Theorem.

Textbook

[1]